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Navier-Stokes System on the Unit Square with no Slip Boundary Condition

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Abstract We consider the two-dimensional Navier-Stokes system on the unit square with no-slip boundary condition. The nonlinear evolution equation for the stream function is studied. Under some hypothesis, we show that the decay of Fourier modes of solutions is power-like.

Keywords Navier-Stokes equations

1 Introduction

In this paper we study Navier-Stokes system for incompressible fluids on the two-dimensional unit square $Q = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. It is written for the velocity vector $u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$, the pressure $p(x, y, t)$ and has the form:

$$\begin{aligned} &\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0, \\ &\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \Delta u_1 - \frac{\partial p}{\partial x}, \\ &\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = \Delta u_2 - \frac{\partial p}{\partial y}. \end{aligned} \tag{1.1}$$

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In (1.1) the viscosity is taken to be 1 and external forcing is absent. The first equation means incompressibility. We assume no slip boundary condition $u(x, y, t) = 0$ for $(x, y) \in \partial Q$.

The system (1.1) is surprisingly difficult. For C^2-boundaries, general results by Ladyzenskaya (see [3]) give the existence of its solutions in the Sobolev space H^2. More recent results related to the square can be found in the papers by Branicki and Moffatt (see [1]), Shankar (see [4]) and Shankar, Kidamki and Mariharan (see [5]). The main feature discovered in [1, 4, 5] is the appearance of vortex patterns and their evolution in time. These patterns can be easily described in the case of periodic boundary conditions and slip boundary conditions when the velocity of the fluid is directed along the boundary (see [2]).

Instead of (1.1) we consider the equivalent equation for the stream function $\psi(x, y, t)$ which is connected with u through the relations $u_1 = -\frac{\partial\psi}{\partial y}$, $u_2 = \frac{\partial\psi}{\partial x}$,

$$\frac{\partial\Delta\psi}{\partial t} - \Delta^2\psi = \frac{\partial\psi}{\partial x}\frac{\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}. \tag{1.2}$$

The equation (1.2) is nothing else but the well-known equation for vorticity $\omega = \Delta\psi$.

Our main concern in this paper is the smoothness of solutions of (1.1) or (1.2). Apply formally Δ^{-1} to both sides of (1.2). It is not so simple step because sometimes Δ^{-1} is not defined. Even if it is defined its form can depend on the space in which Δ^{-1} is considered. We shall discuss the related questions later and now write

$$\frac{\partial\psi}{\partial t} - \Delta\psi = \Delta^{-1}\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi\right). \tag{1.3}$$

The basis of trigonometric functions is not so convenient for us because of the incompressibility equation and the case of the unit square requires a different basis, namely take $\{f_m(x)\}$, $m \ge 1$ consisting of normalized functions defined on [0, 1] for which

$$f_m(0) = f_m'(0) = f_m(1) = f_m'(1) = 0$$

and $\frac{d^4}{dx^4} f_m = \lambda_m f_m$ for some constants $\lambda_m > 0$. In other words, $\{f_m\}$ is the basis of eigenfunctions of the self-adjoint operator $\mathcal{D} = \frac{d^4}{dx^4}$ with aforementioned boundary conditions. These functions are known in elasticity theory[1] but it seems that earlier they did not appear in problems of fluid dynamics.

The advantage of these functions can be seen from the following remark. If

$$\psi(x, y, t) = \sum_{m,n\ge 1} h_{mn}(t) f_m(x) f_n(y) \tag{1.4}$$

is a series in which the coefficients $h_{mn}(t)$ decay fast enough so that the differentiations giving u_1, u_2 are possible then u_1, u_2 always satisfy incompressibility condition and Dirichlet boundary conditions. In other words, only dynamical equation (1.3) remains. A similar basis can be proposed for general domains.

We shall consider the functions ψ given by the series (1.4) and derive an infinite system of ODEs for the coefficients $h_{mn}(t)$. Let us write

$$f_m'' = -m^2(f_m + \xi_m), \quad m \ge 1. \tag{1.5}$$

[1] We thank S.A. Pirogov for this remark.

For large m the functions ξ_m can be considered as small but not very small perturbations of f_m. We have

$$\xi_m = \sum c_{mm'} f_{m'}, \quad c_{mm'} = \langle \xi_m, f_{m'} \rangle.$$

The coefficients $c_{mm'}$ will be analyzed in Sect. 3. For (1.3)

$$\begin{aligned}
\Delta\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \\
&= \sum_{m,n\geq 1} h_{mn}(t)(f''_m \cdot f_n + f_m \cdot f''_n) \\
&= -\sum_{m,n\geq 1}\Bigg((m^2+n^2)h_{mn}(t) + \sum_{m'}(m')^2 c_{m'm} h_{m'n}(t) \\
&\quad + \sum_{n'}(n')^2 c_{n'n} h_{mn'}(t)\Bigg) f_m \cdot f_n.
\end{aligned} \tag{1.6}$$

Thus the LHS of (1.3) takes the form

$$\begin{aligned}
\frac{\partial\psi}{\partial t} - \Delta\psi = \sum_{m,n\geq 1}\Bigg(&\dot{h}_{mn}(t) + (m^2+n^2)h_{mn}(t) \\
&+ \sum_{m'\geq 1}(m')^2 c_{m'm} h_{m'n}(t) + \sum_{n'\geq 1}(n')^2 c_{n'n} h_{mn'}(t)\Bigg) f_m f_n.
\end{aligned} \tag{1.7}$$

Formally the RHS of (1.3) can be also written as a series with respect to $f_m \cdot f_n$

$$\Delta^{-1}\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi\right) = \sum_{m,n\geq 1} N_{mn}(t) f_m f_n. \tag{1.8}$$

The coefficients N_{mn} will be estimated in Sect. 2. Equating the coefficients near each $f_m f_n$, we get the needed system of ODE:

$$\begin{aligned}
&\frac{d}{dt}h_{mn}(t) + (m^2+n^2)h_{mn}(t) + \sum_{m'\geq 1}(m')^2 c_{m'm} h_{m'n}(t) \\
&\quad + \sum_{n'\geq 1}(n')^2 c_{n'n} h_{mn'}(t) = N_{mn}(t).
\end{aligned}$$

Remark that N_{mn} are quadratic functions of h. Now we can formulate the main result of this paper. Assume that $|h_{mn}(0)| \leq \frac{\epsilon}{(m^2+n^2)m^\gamma n^\gamma}$ for all $m \geq 1$, $n \geq 1$, $\frac{1}{2} < \gamma < 1$ and ϵ is sufficiently small.

Main Theorem *If two hypotheses formulated in the text are valid then* $|h_{mn}(t)| \leq \frac{C\epsilon}{(m^2+n^2)m^\gamma \cdot n^\gamma}$ *for all* $t > 0$, $m \geq 1$, $n \geq 1$, *and* C *is an absolute constant.*

The hypotheses in the main theorem can be checked numerically. In Sect. 2 we discuss the properties of the functions f_m, f_n. In Sects. 3, 4 we describe the properties of the Laplacian Δ and its inverse Δ^{-1}. In Sect. 5 we estimate the nonlinear terms in (1.8). The main theorem is proven in Sect. 6.

2 Properties of the Functions f_n

Since $\frac{d^4 f_m}{dx^4} = \lambda_m f_m$ and $\lambda_m > 0$, we can introduce $\tilde{m}$ such that $\lambda_m = \tilde{m}^4$ and write f_m in the form

$$f_m(x) = a\sin\tilde{m}x + b\cos\tilde{m}x + c\sinh\tilde{m}x + d\cosh\tilde{m}x$$

with some coefficients a, b, c, d. Boundary conditions $f_m(0) = f_m'(0) = 0$ give $d = -b$, $c = -a$ and

$$f_m(x) = a(\sin\tilde{m}x - \sinh\tilde{m}x) + b(\cos\tilde{m}x - \cosh\tilde{m}x).$$

From $f(1) = f'(1) = 0$,

$$a(\sin\tilde{m} - \sinh\tilde{m}) + b(\cos\tilde{m} - \cosh\tilde{m}) = 0,$$
$$a(\cos\tilde{m} - \cosh\tilde{m}) - b(\sin\tilde{m} + \sinh\tilde{m}) = 0.$$

This system of linear equations must have a non-zero solution. Therefore its determinant is zero

$$\sin^2\tilde{m} - \sinh^2\tilde{m} + (\cos\tilde{m} - \cosh\tilde{m})^2 = 0$$

which implies

$$\cos\tilde{m}\cdot\cosh\tilde{m} = 1. \tag{2.1}$$

As $\tilde{m}\to\infty$ the second factor $\cosh\tilde{m}$ tends to infinity. Therefore $\cos\tilde{m}\to 0$ and $\tilde{m}$ can be written in the form

$$\tilde{m} = \frac{\pi}{2} + \pi(m-1) + \alpha_m, \tag{2.2}$$

where $m \geq 1$ is an integer and α_m tends to zero exponentially fast. The formula (2.2) explains the meaning of m which was used before. In the bulk of the interval [0, 1] f_m look like the usual trigonometric functions and get distorted near the end-points $x = 0$, $x = 1$.

We shall use mostly

$$f_m^{(1)}(x) = \sin\tilde{m}x - \cos\tilde{m}x + e^{-\tilde{m}x} - \sin\tilde{m}e^{-\tilde{m}(1-x)} + \alpha_m^{(1)}(x), \tag{2.3}$$

where

$$\alpha_m^{(1)}(x) = \frac{\sin\tilde{m} + \cos\tilde{m} - e^{-\tilde{m}}}{\sin\tilde{m} + \sinh\tilde{m}}\left(\cos\tilde{m}x - \frac{1}{2}e^{-\tilde{m}x}\right) - \frac{\cos\tilde{m} - e^{-\tilde{m}}}{\sin\tilde{m} + \sinh\tilde{m}}\cdot\frac{1}{2}\cdot e^{\tilde{m}x}$$
$$+ \frac{2\sin\tilde{m}e^{-\tilde{m}} - e^{-2\tilde{m}}}{\sin\tilde{m} + \sinh\tilde{m}}\cdot\frac{1}{2}\sin\tilde{m}\cdot e^{\tilde{m}x}.$$

By (2.1) it is not difficult to check that $\alpha_m^{(1)}$ are exponentially small. In the remaining part of the paper we shall write "es" to mark exponentially small remainders. The functions $f_m(x)$ differ from the normalized eigen-functions by factors $\mathcal{O}(1)$. In the rest of the paper we shall conveniently make all calculations with $f_m^{(1)}$ given by (2.3). All calculations with the normalized eigen-functions differ by factors $O(1)$. Sometimes we use

$$f_m^{(2)}(x) = \sin\tilde{m}x - \cos\tilde{m}x + e^{-\tilde{m}x} - \sin\tilde{m}e^{-\tilde{m}(1-x)} \tag{2.4}$$

and later make remarks about the remainders.

Write

$$\left(f_m^{(1)}\right)'' = -\tilde{m}^2(f_m^{(1)} + \xi_m) + \alpha_m^{(2)}(x), \tag{2.5}$$

where

$$\xi_m = 2\sin\tilde{m}e^{-\tilde{m}(1-x)} - 2e^{-\tilde{m}x}$$

and $\alpha_m^{(2)}$ are es.

Expand

$$\xi_m = \sum_{m'\geq 1} c_{mm'} f_{m'}^{(1)}.$$

We have

$$\begin{aligned} c_{mm'} &= \langle f_{m'}, \xi_m\rangle \\ &= 2\langle \sin\tilde{m}'x - \cos\tilde{m}'x + e^{-\tilde{m}'x} - \sin\tilde{m}'e^{-\tilde{m}'(1-x)}, \sin\tilde{m}e^{-\tilde{m}(1-x)} - e^{-\tilde{m}x}\rangle \\ &= -2c_{mm'}^{(1)} + \text{es}. \end{aligned} \tag{2.6}$$

Let us write

$$c_{mm'}^{(1)} = \sum_{s=1}^{8} I_s.$$

Here

$$\begin{aligned} I_1 &= \int_0^1 \sin\tilde{m}'xe^{-\tilde{m}x}dx = \frac{\tilde{m}' - e^{-\tilde{m}'}(\tilde{m}'\cos\tilde{m}' + \tilde{m}\sin\tilde{m}')}{\tilde{m}^2 + (\tilde{m}')^2} \\ &= \frac{\tilde{m}'}{\tilde{m}^2 + (\tilde{m}')^2} + \text{es}. \end{aligned}$$

Similarly

$$\begin{aligned} I_2 &= -\int_0^1 \cos\tilde{m}'xe^{-\tilde{m}x}dx = -\frac{\tilde{m} - e^{-\tilde{m}}(\tilde{m}\cos\tilde{m}' - \tilde{m}'\sin\tilde{m}')}{\tilde{m}^2 + (\tilde{m}')^2} \\ &= -\frac{\tilde{m}}{\tilde{m}^2 + (\tilde{m}')^2} + \text{es}. \end{aligned}$$

Also

$$\begin{aligned} I_3 &= \int_0^1 e^{-(\tilde{m}'+\tilde{m})x}dx = \frac{1}{\tilde{m} + \tilde{m}'}(1 - e^{-(\tilde{m}+\tilde{m}')}) \\ &= \frac{1}{\tilde{m} + \tilde{m}'} + \text{es}, \\ I_4 &= -\sin\tilde{m}'\int_0^1 e^{-\tilde{m}'(1-x)}e^{-\tilde{m}x}dx = \text{es}, \\ I_5 &= -\sin\tilde{m}\int_0^1 \sin\tilde{m}'xe^{-\tilde{m}(1-x)}dx \end{aligned}$$

$$= -\frac{\sin\tilde{m}\sin\tilde{m}'\cdot\tilde{m}}{\tilde{m}^2+(\tilde{m}')^2}+\text{es},$$

$$I_6 = \sin\tilde{m}\int_0^1 \cos\tilde{m}'xe^{-\tilde{m}(1-x)}dx$$

$$= \frac{\sin\tilde{m}\cdot\sin\tilde{m}'\cdot\tilde{m}'}{\tilde{m}^2+(\tilde{m}')^2}+\text{es},$$

$$I_7 = -\sin\tilde{m}\int_0^1 e^{-\tilde{m}'x-\tilde{m}(1-x)}dx = \text{es},$$

$$I_8 = \sin\tilde{m}\sin\tilde{m}'\int_0^1 e^{-\tilde{m}'(1-x)-\tilde{m}(1-x)}dx$$

$$= \frac{\sin\tilde{m}\sin\tilde{m}'}{\tilde{m}+\tilde{m}'}+\text{es}.$$

Collecting all the estimates, we obtain in the main order of magnitude,

$$c^{(1)}_{mm'} = \left(\frac{\tilde{m}'}{\tilde{m}^2+(\tilde{m}')^2}-\frac{\tilde{m}}{\tilde{m}^2+(\tilde{m}')^2}+\frac{1}{\tilde{m}+\tilde{m}'}\right)(1+\sin\tilde{m}\cdot\sin\tilde{m}')$$
$$= \frac{2(\tilde{m}')^2}{(\tilde{m}+\tilde{m}')(\tilde{m}^2+(\tilde{m}')^2)}(1+\sin\tilde{m}\cdot\sin\tilde{m}'). \tag{2.7}$$

By (2.6),

$$\langle f_{m'},\xi_m\rangle = -2c^{(1)}_{mm'}+\text{es}$$
$$= -\frac{4(\tilde{m}')^2}{(\tilde{m}+\tilde{m}')(\tilde{m}^2+(\tilde{m}')^2)}(1+\sin\tilde{m}\cdot\sin\tilde{m}')+\text{es}.$$

This is the expression which we shall use in the next section.

3 Laplace Operator Δ

We consider the Hilbert space H whose elements have the form

$$\psi = \sum h_{mn}f_m\cdot f_n,\qquad \sum |h_{mn}|^2<\infty.$$

Take the product $f_m f_n$. As was mentioned above,

$$\Delta(f_m f_n) = -\tilde{m}^2(f_m+\xi_m)f_n-\tilde{n}^2(f_n+\xi_n)f_m$$
$$= -\big((\tilde{m}^2+\tilde{n}^2)f_m\cdot f_n+\tilde{m}^2\xi_m\cdot f_n+\tilde{n}^2 f_m\cdot\xi_n\big).$$

If $\psi=\sum h_{mn}f_m f_n$, then formally

$$\Delta\psi = -\sum_{m,n\geq 1} f_m f_n(\tilde{m}^2+\tilde{n}^2)h_{mn}-\sum_{m,m',n\geq 1} f_m f_n(\tilde{m}')^2\langle\xi_{m'},f_m\rangle\cdot h_{m'n}$$
$$-\sum_{m,n,n'\geq 1} f_m f_n(\tilde{n}')^2\langle\xi_{n'},f_n\rangle h_{mn'}. \tag{3.1}$$

One can take (3.1) as the definition of Laplacian using the $f_m \cdot f_n$ series.

Lemma 3.1 *For any* $\psi = \sum_{m,n} h_{mn} f_m f_n$ *with fast enough decaying* h_{mn},

$$\langle \Delta\psi, \psi\rangle \le 0.$$

Proof From (2.3) and (3.1),

$$\begin{aligned}
\langle \Delta\psi, \psi\rangle &= -\sum_{m,n}\Bigg((\tilde m^2+\tilde n^2)h_{mn}^2 + \sum_{m'}(\tilde m')^2\langle \xi_{m'}, f_m\rangle h_{m'n}h_{mn} \\
&\quad + \sum_{n'}(\tilde n')^2\langle \xi_{n'}, f_n\rangle h_{mn'}h_{mn}\Bigg) \\
&= -\sum_n\sum_m\Bigg(\tilde m^2 h_{mn}^2 + \sum_{m'}\langle \xi_{m'}, f_m\rangle(\tilde m')^2 h_{m'n}h_{mn} + \tilde n^2 h_{mn}^2 \\
&\quad + \sum_{n'}(\tilde n')^2\langle \xi_{n'}, f_n\rangle h_{mn'}h_{mn}\Bigg) \\
&= -\sum_n\sum_m\Bigg(\tilde m^2 h_{mn}^2 + \Bigg\langle \sum_{m'}(\tilde m')^2\xi_{m'}h_{m'n}, f_m\Bigg\rangle\cdot h_{mn} + \tilde n^2 h_{mn}^2 \\
&\quad + \Bigg\langle \sum_{n'}(\tilde n')^2\xi_{n'}h_{mn'}, f_n\Bigg\rangle h_{mn}\Bigg) \\
&= -\sum_n\sum_m\Bigg(\tilde m^2 h_{mn}^2 + \Bigg\langle \sum_{m'}(-f''_{m'} - (\tilde m')^2 f_{m'})h_{m'n}, f_m\Bigg\rangle h_{mn} + \sum_m\sum_n \tilde n^2 h_{mn}^2 \\
&\quad + \Bigg\langle \sum_{n'}(-f''_{n'} - (\tilde n')^2 f_{n'})h_{mn'}, f_n\Bigg\rangle h_{mn}\Bigg) \\
&= \sum_n\sum_m\sum_{m'}\langle f''_m, f_m\rangle h_{m'n}\cdot h_{mn} + \sum_m\sum_n\sum_{n'}\langle f''_{n'}, f_n\rangle h_{mn'}h_{mn} \\
&= -\sum_n\Bigg\langle \sum_{m'}h_{m'n}f'_{m'}, \sum_m h_{mn}f'_m\Bigg\rangle - \sum_m\Bigg\langle \sum_{n'}h_{mn'}f'_{n'}, \sum_{n'}h_{mn'}f'_{n'}\Bigg\rangle.
\end{aligned}$$

The last expression is non-positive. Lemma is proven. □

Denote $h_{mn}^{(1)} = (\tilde m^2+\tilde n^2)h_{mn}$, $\Delta\psi = -\sum g_{mn}f_m f_n$ and write

$$\begin{aligned}
g_{mn} &= h_{mn}^{(1)} + \sum_{m'\ge 1}\frac{(\tilde m')^2\langle \xi_{m'}, f_m\rangle}{(\tilde m')^2+\tilde n^2}h_{m'n}^{(1)} \\
&\quad + \sum_{n'\ge 1}(\tilde n')^2\frac{\langle \xi_{n'}, f_n\rangle}{\tilde m^2+(\tilde n')^2}h_{mn'}^{(1)}.
\end{aligned}\tag{3.2}$$

Using the expression for $\langle \xi_{m'}, f_m\rangle$ and $\langle \xi_{n'}, f_n\rangle$, we can write

$$g = h^{(1)} + K^{(11)}h^{(1)} + K^{(12)}h^{(1)} + K^{(21)}h^{(1)} + K^{(22)}h^{(1)},$$

and $K^{(ij)}$ are linear operators having matrix elements

$$k^{(11)}_{m'm} = -4\frac{(\tilde{m}')^2}{(\tilde{m}')^2+\tilde{n}^2}\cdot\frac{\tilde{m}^2}{(\tilde{m}+\tilde{m}')(\tilde{m}^2+(\tilde{m}')^2)},$$
$$k^{(12)}_{m'm} = \sin\tilde{m}'\cdot\sin\tilde{m}\cdot k^{(11)}_{m'm},$$
$$k^{(21)}_{n'n} = -4\frac{(\tilde{n}')^2}{\tilde{m}^2+(\tilde{n}')^2}\cdot\frac{\tilde{n}^2}{(\tilde{n}'+\tilde{n})((\tilde{n}')^2+\tilde{n}^2)},$$
$$k^{(22)}_{n'n} = \sin\tilde{n}'\cdot\sin\tilde{n}\cdot k^{(21)}_{n'n}.$$

Lemma 3.2 *There exists an absolute constant $C_2 = C_2(\gamma) > 0$ such that if $|h^{(1)}_{mn}| \le \frac{A}{m^\gamma n^\gamma}$ for all $m \ge 1, n \ge 1$ then $|g_{mn}| \le \frac{A\cdot C_2}{m^\gamma n^\gamma}$ for all $m, n \ge 1$.*

Proof We have to estimate the components of each vector $K^{(ij)}h^{(1)}$. We shall do it for $i = j = 1$, other components can be estimated in the same way. We have

$$\sum_{m'\ge 1}\frac{(\tilde{m}')^2}{(\tilde{m}')^2+\tilde{n}^2}\cdot\frac{\tilde{m}^2}{(\tilde{m}'+\tilde{m})((\tilde{m}')^2+(\tilde{m})^2)}\cdot\frac{1}{(\tilde{m}')^\gamma}$$
$$\le \sum_{m'\ge 1}\frac{1}{(\tilde{m}'+\tilde{m})(\tilde{m}')^\gamma}$$
$$= \sum_{m'\le m}\frac{1}{(\tilde{m}'+\tilde{m})(\tilde{m}')^\gamma} + \sum_{m'>m}\frac{1}{(\tilde{m}'+\tilde{m})(\tilde{m}')^\gamma}$$
$$\le \frac{1}{\tilde{m}}\cdot\sum_{m'\le m}\frac{1}{(\tilde{m}')^\gamma} + \sum_{m'>m}\frac{1}{(\tilde{m}')^{\gamma+1}}$$
$$\le C_2\cdot\frac{1}{\tilde{m}^\gamma}.$$

Lemma is proven. □

The lemma shows that the operators $K^{(ij)}$ are bounded operators in the Banach space of sequences $\{h^{(1)}\}$, $|h^{(1)}_{mn}| \le \frac{A}{m^\gamma\cdot n^\gamma}$.

Lemma 3.2 has a stronger version. Take $h^{(1)}_{mn} = \frac{\varphi(\frac{m}{n})}{m^\gamma\cdot n^\gamma}(1+\epsilon(m,n))$ where φ is a bounded Lipschitz function of its argument and $\epsilon(m,n)\to 0$ as $m\to\infty$, $n\to\infty$. Put $t = \frac{\tilde{m}'}{\tilde{m}}$, $dt = \frac{1}{\tilde{m}}$, $u = \frac{\tilde{m}}{\tilde{n}}$. Then

$$\left(K^{(11)}h^{(1)}\right)_{mn} = -4\sum_{m'}\frac{(\tilde{m}')^2}{(\tilde{m}')^2+\tilde{n}^2}\cdot\frac{\tilde{m}^2}{(\tilde{m}+\tilde{m}')((\tilde{m}')^2+(\tilde{m})^2)}h_{\tilde{m}'n}$$
$$= -\frac{4}{\tilde{m}^\gamma\cdot\tilde{n}^\gamma}\cdot\sum_t\frac{t^2}{t^2+(\frac{1}{u})^2}\cdot\frac{1}{1+t}\cdot\frac{1}{1+t^2}\frac{\varphi(tu)(1+\epsilon)}{t^\gamma}dt.$$

Since φ is Lipschitz

$$(K^{(11)}h^{(1)})_{mn} = -\frac{4}{\tilde{m}^\gamma\tilde{n}^\gamma}\int_0^\infty\frac{t^2}{t^2+\frac{1}{u^2}}\cdot\frac{1}{1+t^2}\cdot\frac{1}{1+t}\frac{\varphi(tu)}{t^\gamma}dt\cdot(1+\epsilon_1(m,n)), \quad (3.3)$$

where $\epsilon_1(m,n)$ is a remainder. For the operator $K^{(21)}$ and $s = \frac{\tilde{n}'}{\tilde{n}}$, $ds = \frac{1}{\tilde{n}}$, we have

$$(K^{(21)}h^{(1)})_{mn} = -\frac{4}{\tilde{m}^\gamma \tilde{n}^\gamma}\int_0^\infty \frac{s^2}{s^2+u^2}\cdot\frac{s^2}{1+s^2}\cdot\frac{1}{1+s}\cdot\frac{1}{s^\gamma}\cdot\varphi\left(\frac{u}{s}\right)ds(1+\epsilon_2(m,n)), \quad (3.4)$$

where $\epsilon_2(m,n)$ is another remainder.

Since $K^{(12)}$, $K^{(22)}$ have strongly oscillating terms $\sin\tilde{m}\sin\tilde{m}'$, their contribution has higher order of smallness.

Finally,

$$\Delta = (I+K)\mathcal{D}, \quad (3.5)$$

where $\mathcal{D}h = h^{(1)}$ and is a diagonal operator and $K = K^{(11)} + K^{(21)} + K^{(12)} + K^{(22)}$.

Make the following hypothesis

Hypothesis 1 Consider (3.2) and assume that $|h^{(1)}_{mn}| \le \frac{1}{m^\gamma n^\gamma}$ for all $m, n \ge 1$ and for some m_0, n_0

$$h^{(1)}_{m_0 n_0} = \pm\frac{1}{m_0^\gamma n_0^\gamma}.$$

One can find some positive constant B_1 such that:
If $h^{(1)}_{m_0 n_0} = \frac{1}{m_0^\gamma n_0^\gamma}$, then

$$g_{m_0 n_0} \ge \frac{B_1}{m_0^\gamma n_0^\gamma}.$$

If $h^{(1)}_{m_0 n_0} = -\frac{1}{m_0^\gamma n_0^\gamma}$, then

$$g_{m_0 n_0} \le -\frac{B_1}{m_0^\gamma \cdot n_0^\gamma}.$$

Clearly if $|h^{(1)}_{mn}| \le \frac{A}{m^\gamma \cdot n^\gamma}$ for some number A, then (3.4) is valid with the constant $\frac{B_1}{A}$.

There are some reasons to believe that Hypothesis 1 is valid. Indeed, consider linear heat equation $\frac{\partial\psi}{\partial t} = \Delta\psi$. Eigen-functions of the Laplace operator are $\sin m\pi x \sin n\pi y$ with eigenvalues $-\pi^2(m^2+n^2)$. Therefore all solutions of this equation decay exponentially. Then all solutions written in the basis $f_m \cdot f_n$ should also decay exponentially. This gives some reasons to believe that (3.4) might be true. Certainly an additional numerical checking is needed.

4 The Operator Δ^{-1}

Return back to (3.2). The operator Δ^{-1} shows how to find $h^{(1)}_{mn}$ or h_{mn} knowing g_{mn}. Let us show that this is not always possible. Take $g = \{g_{mn}\}$, $g = \sin m\pi x \sin n\pi y$. It is clear that h corresponds to $\frac{1}{\pi^2(m^2+n^2)}$. It is easy to check that $h^{(1)}_{mn} = O(1)$. Therefore the series in the right-hand side of (3.2) cannot converge.

On the other hand, let $\tilde{K}^{(11)}$, $\tilde{K}^{(21)}$ be the integral operators in the right hand side of (3.3), (3.4), i.e. we neglect the remainders. Then after the change of variable in the first integral

$tu=\lambda$, $dt=\frac{d\lambda}{u}$ and $s=\frac{u}{\lambda}$, $ds=-\frac{ud\lambda}{\lambda^2}$ in the second integral, the operator $\tilde{K}$ takes the form

$$(\tilde{K}\varphi)(u) = -4\int_0^\infty \Big(\frac{\lambda^2}{1+\lambda^2}\cdot\frac{u}{u+\lambda}\cdot\frac{u^{1+\gamma}}{u^2+\lambda^2}\cdot\frac{1}{\lambda^\gamma} + \frac{1}{1+\lambda^2}\cdot\frac{\lambda}{u+\lambda}\cdot\frac{u^2}{u^2+\lambda^2}\cdot\frac{\lambda^\gamma}{u^\gamma}\cdot\frac{u}{\lambda^2}\Big)\varphi(\lambda)d\lambda. \tag{4.1}$$

We shall prove the following lemma.

Lemma 4.1 (Invertibility of $(I+\tilde{K})$ in L^∞) *Let $\gamma=1/2$. Then the operator $(I_d+\tilde{K})$ maps L^∞ to L^∞ and has a bounded inverse.*

Proof Write the kernel $\tilde{K}$ as

$$\begin{aligned}(\tilde{K}\phi)(u) &= -4\int_0^\infty \Big(\frac{\lambda^2u^2}{1+\lambda^2u^2}\cdot\frac{1}{1+\lambda}\cdot\frac{1}{1+\lambda^2}\cdot\frac{1}{\lambda^\gamma} + \frac{1}{1+\lambda^2u^2}\cdot\frac{\lambda}{1+\lambda}\cdot\frac{1}{1+\lambda^2}\cdot\frac{\lambda^\gamma}{\lambda^2}\Big)\phi(\lambda u)d\lambda\\ &= -4\int_0^\infty \frac{\lambda^2u^2+\lambda^{2\gamma-1}}{(1+\lambda^2u^2)(1+\lambda)(1+\lambda^2)}\cdot\lambda^{-\gamma}\phi(\lambda u)d\lambda,\end{aligned}$$

where in the second part of the expression for $\tilde{K}$ we made the change of variable $\lambda\to\lambda u$. If $\gamma=\frac{1}{2}$, then

$$(\tilde{K}\phi)(u) = -4\int_0^\infty \frac{1}{(1+\lambda)(1+\lambda^2)}\cdot\lambda^{-\frac{1}{2}}\phi(\lambda u)d\lambda.$$

Denote $u=e^t$, $\lambda=e^{-\tau}$, and $\tilde{\phi}(t)=\phi(e^t)$. Then in the new variables the operator $\tilde{K}$ becomes the operator $M: L^\infty\to L^\infty$ acting by the formula:

$$\big(M(\tilde{\phi})\big)(t) = (K\phi)(e^t),$$

and

$$\begin{aligned}(M\tilde{\phi})(t) &= -4\int_{-\infty}^\infty \frac{e^{-\frac{1}{2}\tau}}{(1+e^{-\tau})(1+e^{-2\tau})}\tilde{\phi}(t-\tau)d\tau\\ &=: \int_{-\infty}^\infty m(\tau)\tilde{\phi}(t-\tau)d\tau,\end{aligned}$$

i.e. M is the convolution. The Fourier Transform of the kernel m is given by

$$\hat{m}(k) = -4\int_{-\infty}^\infty \frac{e^{-\frac{1}{2}\tau}}{(1+e^{-\tau})(1+e^{-2\tau})}e^{-i\tau k}d\tau.$$

It is easy to check that $\hat{m}(k)$ is a Schwartz function of k. The numerics show that $\tilde{m}(k)\neq -1$. Therefore the operator $(I+M)$ is invertible in the space L^∞. □

Now we can formulate our second hypothesis.

Hypothesis 2 The operator $(I+K)$ in (3.5) has a bounded inverse in L^∞.

5 The Estimate of the Nonlinear Terms

In this section we estimate

$$\begin{aligned}
&\Delta^{-1}\left(\frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial y}\Delta\psi-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi\right)\\
&\quad=\Delta^{-1}\left(\frac{\partial}{\partial y}\cdot\left(\frac{\partial\psi}{\partial x}\cdot\Delta\psi\right)-\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\cdot\Delta\psi\right)\right)\\
&\quad=\frac{\partial}{\partial y}\left(\Delta^{-1}\left(\frac{\partial\psi}{\partial x}\cdot\Delta\psi\right)-\frac{\partial}{\partial x}\left(\Delta^{-1}\left(\frac{\partial\psi}{\partial y}\Delta\psi\right)\right)\right).
\end{aligned}\tag{5.1}$$

We write

$$\begin{aligned}
\psi&=\sum_{m,n}h_{mn}f_m\cdot f_n,\\
\Delta\psi&=\sum_{m,n}g_{mn}f_m\cdot f_n.
\end{aligned}$$

If

$$|h_{mn}|\le\frac{A}{(m^2+n^2)m^\gamma\cdot n^\gamma}$$

for some constant A, then from Lemma 3.2,

$$|g_{mn}|\le\frac{C_2A}{m^\gamma n^\gamma}.$$

Formally

$$\begin{aligned}
\frac{\partial\psi}{\partial x}&=\sum_{m,n\ge1}h_{mn}f'_m\cdot f_n\\
&=\sum_{m,n\ge1}h_{mn}m\cdot\tilde f_m\cdot f_n.
\end{aligned}$$

It is easy to check that $\|\tilde f_m\|=\mathcal{O}(1)$ and $\tilde f_m$ is a linear combination of $e^{\pm i\tilde m x}, e^{\pm\tilde m x}$. Let us write

$$\begin{aligned}
\tilde f_{m_1}\cdot f_{m_2}&=\sum_m d^{(m)}_{m_1,m_2}f_m,\\
f_{n_1}\cdot f_{n_2}&=\sum_n d^{(n)}_{n_1,n_2}f_n.
\end{aligned}$$

Then

$$\begin{aligned}
|d^{(m)}_{m_1,m_2}|&\le\max_{\pm}\frac{\text{const}}{|m_1\pm m_2\pm m|+1},\\
|d^{(n)}_{n_1,n_2}|&\le\max_{\pm}\frac{\text{const}}{|n_1\pm n_2\pm n|+1}.
\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial\psi}{\partial x}\cdot\Delta\psi &= \sum_{\substack{m_1,n_1\\m_2,n_2}} h_{m_1,n_1}\cdot m_1\tilde{f}_{m_1}\cdot f_{n_1}g_{m_2,n_2}f_{m_2}\cdot f_{n_2}\\ &= \sum_{m,n\geq 1} f_m f_n\cdot\sum_{\substack{m_1,n_1\\m_2,n_2}} h_{m_1,n_1}m_1\cdot d^{(m)}_{m_1,m_2}\cdot g_{m_2,n_2}d^{(n)}_{n_1,n_2}\\ &= \sum_{m,n\geq 1} f_m\cdot f_n\sum_{m_1\geq 1}\sum_{n_1\geq 1}\sum_{m_2\geq 1}\sum_{n_2\geq 1} h_{m_1,n_1}\cdot m_1\cdot d^{(m)}_{m_1,m_2}\cdot g_{m_2,n_2}d^{(n)}_{n_1,n_2}, \qquad (5.2)\end{aligned}$$

it is enough to estimate

$$\begin{aligned}&\sum_{m_1\geq 1}\sum_{n_1\geq 1}\sum_{m_2\geq 1}\sum_{n_2\geq 1}\frac{m_1}{(m_1^2+n_1^2)m_1^\gamma\cdot n_1^\gamma}\\ &\quad\times\frac{1}{|m_1\pm m_2\pm m|+1}\cdot\frac{1}{m_2^\gamma\cdot n_2^\gamma}\cdot\frac{1}{|n_1\pm n_2\pm n|+1}. \qquad (5.3)\end{aligned}$$

Consider

$$\sum_{n_2}\frac{1}{n_2^\gamma\cdot(|n\pm n_1\pm n_2|+1)}.$$

Decompose the sum into three parts: $n_2\leq\frac{1}{2}|n\pm n_1|$, $\frac{1}{2}|n\pm n_1|\leq n_2\leq 2|n\pm n_1|$, and $n_2\geq 2|n\pm n_1|$. The first part is not more than $\frac{\text{const}}{|n\pm n_1|^\gamma+1}$, the second part is not more than $\frac{\text{const}\cdot\ln(|n\pm n_1|+1)}{|n\pm n_1|^\gamma+1}$, and the third part is not more than $\frac{\text{const}}{|n\pm n_1|^\gamma+1}$. Therefore

$$\sum_{n_2}\frac{1}{n_2^\gamma(|n\pm n_1\pm n_2|+1)}\leq\frac{\text{const}\cdot\ln(|n\pm n_1|+1)}{|n_1\pm n|^\gamma+1}.$$

In the same way

$$\sum_{m_2\geq 1}\frac{1}{m_2^\gamma(|m\pm m_1\pm m_2|+1)}\leq\frac{\text{const}\cdot\ln(|m\pm m_1|+1)}{|m_1\pm m|^\gamma+1}.$$

Consider the sum

$$\begin{aligned}&\sum_{n_1}\frac{\ln(|n_1\pm n|+1)}{(m_1^2+n_1^2)n_1^\gamma(|n\pm n_1^\gamma|+1)}\\ &\quad=\sum_{n_1\leq m_1}\frac{\ln(|n_1\pm n|+1)}{(m_1^2+n_1^2)n_1^\gamma\cdot(|n_1\pm n|^\gamma+1)}\\ &\quad\quad+\sum_{n_1>m_1}\frac{\ln(|n_1\pm n|+1)}{(m_1^2+n_1^2)n_1^\gamma\cdot(|n_1\pm n|^\gamma+1)}. \qquad (5.4)\end{aligned}$$

For the first sum we can write

$$\sum_{n_1\leq m_1}\frac{\ln(|n_1\pm n|+1)}{(m_1^2+n_1^2)n_1^\gamma(|n_1\pm n|^\gamma+1)}\leq\frac{1}{m_1^2}\sum_{n_1\leq m_1}\frac{\ln(|n_1\pm n|+1)}{n_1^\gamma\cdot(|n_1\pm n|^\gamma+1)}.$$

Case 1. $m_1 \le \frac{n}{2}$. In this case $(|n_1 \pm n|^\gamma + 1) \ge \text{const} \cdot n^\gamma$,

$$\sum_{n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n|^\gamma + 1)} \le \frac{\text{const} \cdot \ln n}{n^\gamma} \cdot \sum_{n_1 \le m_1} \frac{1}{n_1^\gamma} = \frac{\text{const} \cdot \ln n \cdot m_1^{1-\gamma}}{n^\gamma}$$

and

$$\frac{1}{m_1^2} \sum_{n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n|^\gamma + 1)} \le \frac{\text{const} \cdot \ln n}{m_1^{1+\gamma} \cdot n^\gamma}.$$

Case 2. $\frac{n}{2} < m_1 \le 2n$. In this case

$$\begin{aligned}
&\frac{1}{m_1^2} \sum_{n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n|^\gamma + 1)} \\
&\quad \le \frac{1}{m_1^2} \left(\sum_{n_1 \le \frac{m_1}{4}} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma \cdot (|n_1 \pm n|^\gamma + 1)} + \sum_{\frac{m_1}{4} \le n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n|^\gamma + 1)} \right) \\
&\quad \le \frac{1}{m_1^2} \left[\frac{\text{const} \cdot \ln n}{n^\gamma \cdot m_1^{\gamma-1}} + \frac{\text{const} \cdot \ln n}{m_1^{\gamma+\gamma-1}} \right] \\
&\quad \le \frac{\text{const} \cdot \ln n}{m_1^2} \left[\frac{1}{n^\gamma \cdot m^{\gamma-1}} + \frac{1}{m_1^{2\gamma-1}} \right] \\
&\quad \le \frac{\text{const} \cdot \ln n}{n^\gamma \cdot m_1^{\gamma+1}}.
\end{aligned}$$

Case 3. $m_1 > 2n$. Now

$$\begin{aligned}
&\sum_{n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n| + 1)} \\
&\quad = \left(\sum_{n_1 \le \frac{n}{2}} + \sum_{\frac{n}{2} < n_1 \le 2n} + \sum_{2n < n_1 < m_1} \right) \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma \cdot (|n_1 \pm n|^\gamma + 1)} \\
&\quad \le \text{const} \cdot \left(\frac{(\ln n) \cdot n^{1-\gamma}}{n^\gamma} + \frac{1}{n^\gamma} \frac{\ln n}{n^{\gamma-1}} + \frac{\ln n}{n^{2\gamma-1}} \right) \\
&\quad \le \frac{\text{const} \cdot \ln n}{n^{2\gamma-1}}
\end{aligned}$$

and

$$\frac{1}{m_1^2} \sum_{n_1 \le m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^\gamma (|n_1 \pm n|^\gamma + 1)} \le \frac{\text{const} \cdot \ln n}{m_1^{\gamma+1} \cdot n^\gamma}.$$

Now we shall consider

$$\sum_{n_1 > m_1} \frac{\ln(|n_1 \pm n| + 1)}{(m_1^2 + n_1^2) \cdot n_1^\gamma \cdot (|n_1 \pm n|^\gamma + 1)} \le \sum_{n_1 > m_1} \frac{\ln(|n_1 \pm n| + 1)}{n_1^{2+\gamma} (|n_1 \pm n|^\gamma + 1)}.$$

Case 1. $m_1 < \frac{n}{2}$. In this case

$$\begin{aligned}
&\sum_{m_1<n_1<\frac{3}{4}n} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} + \sum_{\frac{3}{4}n\le n_1<2n} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} \\
&\quad + \sum_{n_1\ge 2n} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} \\
&\quad \le \frac{\mathrm{const}\cdot\ln n}{n^\gamma}\cdot\frac{1}{m_1^{\gamma+1}} + \frac{\mathrm{const}\cdot\ln n}{n^{\gamma-1}\cdot n^{2+\gamma}} + \frac{\mathrm{const}\cdot\ln n}{n^{1+2\gamma}} \\
&\quad \le \frac{\mathrm{const}\cdot\ln n}{m_1^{\gamma+1}n^\gamma} + \frac{\mathrm{const}\cdot\ln n}{m_1^{1+\gamma}n^\gamma} + \frac{\mathrm{const}\cdot\ln n}{m_1^{1+\gamma}n^\gamma} \\
&\quad = \frac{\mathrm{const}\cdot\ln n}{m_1^{1+\gamma}n^\gamma}.
\end{aligned}$$

Case 2. $\frac{n}{2} \le m_1 < 2n$. In this case

$$\begin{aligned}
&\sum_{n_1>m_1} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} \\
&\quad = \sum_{m_1<n_1<2n} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} + \sum_{n_1\ge 2n} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} \\
&\quad \le \frac{\mathrm{const}\cdot\ln n}{m_1^{2+\gamma}\cdot n^{\gamma-1}} + \frac{\mathrm{const}\cdot\ln n}{n^{1+2\gamma}} \\
&\quad \le \frac{\mathrm{const}\cdot\ln n}{m_1^{1+\gamma}\cdot n^\gamma}.
\end{aligned}$$

Case 3. $m_1 > 2n$. In this case

$$\begin{aligned}
\sum_{n_1\ge m_1} \frac{\ln(|n_1 \pm n|+1)}{n_1^{2+\gamma}(|n_1 \pm n|^\gamma+1)} &\le \mathrm{const}\sum_{n_1\ge m_1}\frac{\ln n}{n_1^{2+2\gamma}} \le \mathrm{const}\frac{\ln m_1}{m_1^{1+2\gamma}} \\
&= \mathrm{const}\frac{1}{m_1^{1+\gamma}}\cdot\frac{\ln m_1}{m_1^\gamma} \le \mathrm{const}\frac{1}{m_1^{1+\gamma}}\cdot\frac{\ln n}{n^\gamma}.
\end{aligned}$$

The conclusion which follows from all these inequalities gives

$$\sum_{n_1} \frac{\ln(|n_1 \pm n|+1)}{(m_1^2+n_1^2)n_1^\gamma(|n \pm n_1^\gamma|+1)} \le \frac{\mathrm{const}\cdot\ln n}{m_1^{1+\gamma}\cdot n^\gamma}. \tag{5.5}$$

It remains to consider the summation over m_1:

$$\sum_{m_1\ge 1} \frac{m_1^{1-\gamma}\ln(|m \pm m_1|+1)}{m_1^{1+\gamma}(|m_1 \pm m|^\gamma+1)}\cdot\frac{\ln n}{n^\gamma} = \sum_{m_1\ge 1}\frac{\ln(|m \pm m_1|+1)}{m_1^{2\gamma}(|m_1 \pm m|^\gamma+1)}\cdot\frac{\ln n}{n^\gamma}.$$

The same arguments as before show that

$$\sum_{m_1\geq 1}\frac{\ln(|m\pm m_1|+1)}{m_1^{2\gamma}(|m_1\pm m|^\gamma+1)}=\left(\sum_{m_1\leq\frac{m}{2}}+\sum_{\frac{m}{2}<m_1\leq 2m}+\sum_{m_1>2m}\right)\frac{\ln(|m\pm m_1|+1)}{m_1^{2\gamma}(|m_1\pm m|^\gamma+1)}$$

$$\leq\frac{\text{const}\ln m}{m^{3\gamma-1}}\leq\frac{\text{const}}{m^\gamma}$$

if $\gamma\geq\frac{1}{2}$. Returning back to (5.2) we conclude that

$$\frac{\partial\psi}{\partial x}\cdot\Delta\psi=\sum I_{mn}f_m f_n$$

and

$$|I_{mn}|\leq\frac{\text{const}\cdot\ln n}{m^\gamma\cdot n^\gamma}. \tag{5.6}$$

Now we consider $\Delta^{-1}(\frac{\partial\psi}{\partial x}\Delta\psi)$. Since $\Delta=(I+K)\mathcal{D}$ (see (3.5)), we have $\Delta^{-1}=\mathcal{D}^{-1}(I+K)^{-1}$ and

$$\Delta^{-1}\left(\frac{\partial\psi}{\partial x}\Delta\psi\right)=\sum J_{mn}f_m\cdot f_n, \tag{5.7}$$

where

$$|J_{mn}|\leq\frac{\text{const}\cdot\ln n}{m^\gamma\cdot n^\gamma(m^2+n^2)}. \tag{5.8}$$

In view of (5.7) and (5.8), we can differentiate (5.7) with respect to y:

$$\begin{aligned}\frac{\partial}{\partial y}\left(\Delta^{-1}\frac{\partial\psi}{\partial x}\Delta\psi\right)&=\sum_{m,n'\geq 1}J_{mn'}f_m\cdot f'_{n'}\\&=\sum_{m,n'\geq 1}J_{mn'}n'\cdot f_m\cdot\tilde{f}_{n'}\\&=\sum_{m,n\geq 1}f_m\cdot f_n\cdot\sum_{n'\geq 1}J_{mn'}\cdot n'd_{n'}^{(n)}\end{aligned}$$

and $|d_{n'}^{(n)}|=|\langle f_{n'},f_n\rangle|\leq\frac{\text{const}}{|n\pm n'|+1}$. Then as before

$$\begin{aligned}\left|\sum_{n'}J_{mn'}\cdot n'\cdot d_{n'}^{(n)}\right|&\leq\frac{\text{const}}{m^\gamma}\sum_{n'}\frac{\ln n'}{(n')^{\gamma-1}(m^2+(n')^2)(|n\pm n'|+1)}\\&=\frac{\text{const}}{m^\gamma}(\Sigma_1+\Sigma_2+\Sigma_3)\end{aligned}$$

and

$$\begin{aligned}\Sigma_1&=\sum_{n'\leq\frac{n}{2}}\frac{\ln n'}{(n')^{\gamma-1}(m^2+(n')^2)(|n\pm n'|+1)}\\&\leq\frac{\text{const}\cdot\ln n}{n}\sum_{n'\leq\frac{n}{2}}\frac{1}{(n')^{\gamma+1}}\leq\frac{\text{const}\cdot\ln n}{n};\end{aligned}$$

$$
\begin{aligned}
\Sigma_2 &= \sum_{\frac{n}{2}<n'\le n} \frac{\ln n'}{(n')^{\gamma-1}(m^2+(n')^2)(|n\pm n'|+1)} \\
&\le \frac{\text{const}}{n^{\gamma+1}} \cdot \sum_{\frac{n}{2}<n'\le n} \frac{\ln n'}{|n\pm n'|+1} \\
&\le \frac{\text{const}\cdot \ln^2 n}{n^{\gamma+1}}; \\
\Sigma_3 &= \sum_{n'>n} \frac{\ln n'}{(n')^{\gamma-1}(m^2+(n')^2)(|n\pm n'|+1)} \\
&\le \text{const} \sum_{n'>n} \frac{\ln n'}{(n')^{2+\gamma}} \\
&\le \frac{\text{const}\cdot \ln n}{n^{1+\gamma}}.
\end{aligned}
$$

All these estimates are good for us. Therefore for $\psi = \sum_{m,n\ge 1} h_{mn} f_m f_n$ with

$$|h_{mn}| \le \frac{A}{(m^2+n^2)m^\gamma \cdot n^\gamma},$$

we have the estimate for $N(\psi)$ (see (1.8)) with

$$|N_{mn}| \le \frac{C\cdot A^2}{m^\gamma n^\gamma}, \tag{5.9}$$

and C is a constant independent of A.

6 The Proof of the Main Theorem (Trapping Argument)

Assume that at $t=0$ the coefficients $h_{mn}(0)$ satisfy the inequalities

$$|h_{mn}(0)| < \frac{\epsilon}{(m^2+n^2)m^\gamma n^\gamma}$$

and ϵ is sufficiently small. We must prove that for all $t>0$,

$$|h_{mn}(t)| < \frac{C\epsilon}{(m^2+n^2)m^\gamma n^\gamma}, \quad m\ge 1, n\ge 1,$$

where C is some absolute constant.

Indeed if this is wrong, then there are t_0, m_0, n_0 such that

$$|h_{mn}(t)| \le \frac{C\epsilon}{(m^2+n^2)m^\gamma n^\gamma}, \quad \text{for all } m\ge 1,\ n\ge 1,$$

and

$$|h_{mn}(t_0)| = \frac{C\epsilon}{(m_0^2+n_0^2)m_0^\gamma \cdot n_0^\gamma}.$$

We can take t_0 to be minimal for which this conditions holds. Consider the case

$$h_{m_0 n_0}(t_0) = \frac{C\epsilon}{(m_0^2 + n_0^2) m_0^\gamma n_0^\gamma}, \tag{6.1}$$

the other case is considered in the same way. Equation (6.1) implies that $h_{mn}(t_0) = \max_{0 \le t \le t_0} h_{mn}(t)$. Let us show that this is impossible. It follows from Hypothesis 1 that

$$(\Delta\psi)_{m_0,n_0}(t_0) \le -\frac{B_1 \epsilon}{(m_0^2 + n_0^2) m_0^\gamma n_0^\gamma}.$$

On the other hand, from (5.9) it follows that the nonlinear terms in (1.7) for m_0, n_0 have absolute values less that $\frac{\text{const}\cdot\epsilon^2}{m_0^\gamma n_0^\gamma}$. If ϵ is small enough, the derivative $\frac{dh_{m_0 n_0}}{dt_0}$ is negative and $h_{m_0 n_0}$ at $t = t_0$ cannot have maximum.

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